End extensions of models of fragments of PA

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$I\Sigma_n$: induction for Σ_n formulas (plus base theory)

$B\Sigma_n$: $I\Delta_0$ + collection for Σ_n formulas

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Theorem (MacDowell-Specker, 1961)

Every model of PA has a proper elementary end extension.

Theorem (Paris-Kirby, 1978)

For any $n \geq 2$, if M is a countable model of $B\Sigma_n$, then M has a proper Σ_n-elementary end extension $K \models I\Delta_0$.

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The Kirby-Paris construction used very strongly the countability of the model. In view of the cardinality-free statement of the MacDowell-Specker Theorem, we might expect the conclusion of Theorem 1 to hold for models of any cardinality. Such a possibility was first suggested by A. Wilkie.

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Theorem (Clote, 1986/1998)

For any $n \geq 2$, if M is a model of $I\Sigma_n$, then M has a proper Σ_n-elementary end extension $K \models I\Delta_0$.

P. Clote and J. Krajiček. Open problems. *Oxford Logic Guides*, volume 23, *Arithmetic, proof theory and computational complexity* (Prague, 1991). Oxford University Press, New York, 1993.

Problem 1 (Fundamental problem F). Does every countable model of $B\Sigma_1$ have a proper end extension $K \models I\Delta_0$?

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Proofs of the Paris-Kirby and Clote results based on restricted ultrapower constructions

A. J. Wilkie and J. B. Paris. On the existence of end extensions of models of bounded induction. In *Logic, methodology and philosophy of science, VIII (Moscow, 1987)*, volume 126 of *Stud. Logic Found. Math.*, 143–161, North-Holland, 1989.

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$I\Delta_0$-fullness: saturation condition

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$I\Delta_0$-fullness: saturation condition

Theorem (Wilkie-Paris, 1989)

For every countable $M \models B\Sigma_1$, if M is $I\Delta_0$-full, then there exists K such that $M \subset_e K \models I\Delta_0$.

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5 natural conditions, each of which implies $I\Delta_0$-fullness
the most natural one: *exp*

REMARK. A direct proof that any countable model of $B\Sigma_1$ which is closed under exponentiation has a proper end extension to a model of $I\Delta_0$ may be obtained by mimicking the proof of Theorem 4 but with "Semantic Tableaux consistency of Γ" in place of "Γ-full" and adding a new constant symbol $\pi > M$ to ensure that the end extension is proper.

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Our approach combines
(a) well-known procedure of extending a consistent theory to a maximal consistent one
(b) consideration of structures whose universes are sets of definable elements.

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A. Enayat and T. L. Wong. Unifying the model theory of first-order and second-order arithmetic via WKL_0^*. *Ann. Pure Appl. Logic* 168 (2017), 1247–1252.

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Theorem (Slaman). $I\Delta_1 + exp \Rightarrow B\Sigma_1$.